SIAM REVIEW
Vol. 17, No. 3, July 1975

EDUCATION IN APPLIED MATHEMATICS

THE MATHEMATICAL TRAINING OF THE NONACADEMIC MATHEMATICIAN*

Abstract. This paper contains the proceedings of a symposium on "The Mathematical Training of the Nonacademic Mathematician" held at Rensselaer Polytechnic Institute on May 3, 1974. The statements presented by the moderator and invited panelists are included, as well as a summary of the subsequent discussion.

1. Introduction. On May 3, 1974, the Department of Mathematical Sciences at Rensselaer Polytechnic Institute held a symposium on "The Mathematical Training of the Nonacademic Mathematician". The symposium was organized by Professor Richard C. DiPrima, and the speakers were:

Moderator: GEORGE H. HANDELMAN, Rensselaer Polytechnic Institute,
Panelists: BURTON H. COLVIN, National Bureau of Standards,
EDWARD W. HART, General Electric Research and Development Center,
ALAN J. HOFFMAN, IBM Corporation,
MURRAY S. KLAMKIN, Ford Motor Company (now at University of Waterloo),
J. J. SEIDEL, Technological University of Eindhoven, the Netherlands.

Approximately 125 persons were in attendance, representing more than 25 institutions. The speakers presented brief statements of their views on the subject of the symposium, and this was followed by a general discussion involving many members of the audience. The remarks of the moderator and panelists were transcribed and later edited for publication by the speakers. They appear below, together with a summary of the discussion prepared by Professor William E. Boyce.

Partial financial support for the symposium was provided by the Mathematics Branch of the Office of Naval Research.

2. Speakers' remarks.

George Handelman. I thought that in serving as moderator of the panel, it might be appropriate to say a few words about the past, as I see it, in terms of education in the use of mathematics. Going back to the late 1930's and early 1940's, one can review the article[1] by Thornton Fry, which has become a sort of bible for people interested in applications and the needs for mathematicians in industry. I think it is safe to say that during those days, barring those scientists working in statistical applications and actuarial mathematics, the number of mathematicians working in industry was exceedingly limited. I suspect that in my graduate school days I had the pleasure of seeing and listening to most of the well-known industrial mathematicians. The fact that I can name almost all of them and certainly cannot

* Received by the editors September 19, 1974.

[1] Thornton C. Fry, *Industrial mathematics*, Research—A National Resource, vol. 2, U.S. House of Representatives Doc., 77th Congress, 1940.

even begin to name more than a fraction of those in similar positions today shows that a marked change has taken place. In fact, we can almost tick them off: Thornton Fry, Henry Bode and Sergei Shelkunoff at Bell Telephone Laboratories; Hillel Poritsky and Gabriel Kron at General Electric; Joseph Slepian at the Westinghouse Research Laboratories; and Theodore Theodorsen and Nicholas Minorsky working in government laboratories.

In addition, the chief centers educating mathematicians to work in nonacademic professions prior to 1940–41 were mainly in the departments of aeronautics or mechanics or engineering physics and the like. The big development in education of applied mathematicians per se started in this country in late 1940–41 with the growth of groups at Brown University and New York University. These were primarily the products of two very strong individuals, Richard Courant at N.Y.U. and R. G. D. Richardson at Brown. It is interesting to note that these programs were originally started as educational programs; they were initiated as programs to develop mathematicians who could work in government and industry on defense and related problems. In fact, a great deal of the funding which started these programs came from the Federal Government—ESMWT, the Emergency Science Manpower War Training Program. At that time, then, the primary emphasis was on the education and development of mathematicians who would be able to deal with problems which came from the real world.

In many ways, life was somewhat simpler then because the type of material which one had to cover was reasonably well delineated. As far as mathematics itself was concerned, one could list such areas as classical analysis and algebra, differential equations, integral equations, numerical and graphical analysis and some work in statistics. The areas of applications were largely concentrated in fluid mechanics, solid mechanics, wave motion, vibrations and electromagnetic theory. It is interesting to note that the program at Brown was not just called applied mathematics, but went by the title, "Program for Advanced Training in Mechanics and Applied Mathematics"; the popular name for the group at that time was "The Mechanics School". The choice of name probably had an unplanned public relations value since it meant that the layman knew that the participants were doing practical things—obviously they could fix automobiles. Of course, very few of us in the program could afford automobiles; and if we could, it is questionable whether we could fix them. In terms of what one hoped to accomplish, the program was reasonably successful in that many of the participants received an education which lent itself to problems of the day. An additional feature which should be mentioned is that almost all of the teachers in these programs were educated outside of this country.

It would be pleasant to say that we have just reviewed current events; but, unfortunately, it may be closer to medieval history. Perhaps, we should use Marlowe's line,[2] "but that was in another country"; hopefully, his next line does not apply. Nevertheless, the situation has changed radically since that time. In the first place, we have produced several generations of applied mathematicians who have been educated in this country. At some universities, the applied mathematician is almost a respected member of the academic community; and this has

[2] Christopher Marlowe, *The Jew of Malta*, Act IV, Scene 1.

some dangers attached to it. I think that as we become more and more received within the halls of ivy, there is a danger that we can concentrate on developing our younger colleagues primarily for academic careers, and the service aspect, originally thought of many years ago, is gradually being neglected. This is one very important reason for a meeting of this sort.

We have also done something else which I think is both good and bad at the same time. We have successfully educated, as far as mathematical techniques are concerned, many students outside the field of mathematics. We have done such a magnificent job at times that many of these nonmathematicians are taking positions that might be held by mathematicians. This is something which should make us both happy and sad.

It also goes without saying that there has been a vast increase in the areas of mathematics which are now applicable, and an even broader increase in the breadth, scope and type of problems to which mathematics can be applied.

All of this brings to mind a number of questions which may be rather trivial and which our guests may be able to dismiss quite quickly. Let me raise them nevertheless. I think it is quite possible that each one of us could design a curriculum which would reproduce ourselves, and if we were even a bit wiser, could reproduce another colleague or two. On the surface, this may appear rather good. However, in terms of recent history, which has shown such a vast change in the forms of mathematics being used and the areas to which mathematics can be applied, and with the consideration that we are trying to educate today people who will be practitioners for the next thirty-five or forty years, there is a serious question as to whether such a program of education will be sufficiently good.

Another question which bothers me is how we can be assured that our students of today will be able to recognize and to learn new areas of mathematics which are needed for future applications. This is not an easy problem but is probably somewhat simpler than the following. This question stems from the fact that one can only do respectable work in applications if one has a thorough understanding of the areas to which mathematics is to be applied. What can we do in our education today to insure that our students in the future will be able to learn new areas of application as the need arises?

Finally, I think we should remember that the applied mathematician, working in the nonacademic milieu, must be able to understand the problems of the nonprofessional mathematician, and, having developed a solution, must be able to put it in terms which will be understandable and useful to the customer who is not a professional mathematician. Thus, what we really have is a problem in the teaching of communication by reading, writing and listening. I am not sure how one can accomplish this, but it must be done.

I think our distinguished guests of today will be able to answer all these questions for us and probably propose many more interesting queries as well.

Burton H. Colvin. Before discussing the training of mathematicians for the nonacademic world, let me ask you to consider what it is that mathematically trained individuals do to earn a living outside the academic community. People are always asking, "What do mathematicians do?" During World War II, I worked with the National Defense Research Committee, and it was a great pleasure to have an occasional opportunity to chat with Warren Weaver. I remember Warren telling on one occasion how the neighbors were always asking Mrs. Weaver what Warren did for a living. Mrs. Weaver would reply with considerable pride, "Why, he is a mathematician," and they would invariably respond, "What else does he do?" It is this "what else" that nonacademic mathematicians do that I want to talk about.

To suggest the role that mathematicians play in industry, in government, and elsewhere outside academia, let me try to phrase a few questions which characterize their activities. Here are four typical questions:

(i) Can X be modeled?

(ii) Is X correct?

(iii) Is X possible?

(iv) Can X be optimized?

I ask you to interpret these questions freely, unfettering your imagination to roam as far afield as you wish in defining X. Think of X, perhaps, as a problem in the physical sciences or in chemistry or in engineering, an experiment in the laboratory or in outer space, possibly an engineering systems study, a study in health care or maybe a project in economics or biology or medicine. Very often, X is just a feeling of uneasiness, a sort of visceral queasiness that "something is not right."

By "modeling", I mean representing X in some acceptably realistic way using appropriate mathematics—subject, of course, to rational constraints on manpower, time, computer expense, data requirements, etc. A good model provides a reasonable representation of the problem or experiment or phenomenon X we are talking about and can be used (perhaps through a sequence of studies or analyses or computations or refinements) to solve the problem, refine the experiment, optimize the system, or provide some insight as the basis for policy decision.

Is X indeed correct? I mean to suggest here not merely the determination of a simple technical question, but more complex questions of approximation, accuracy, precision, and of statistical analysis. Even more generally, the question may be whether a large scale systems model in engineering or economics or operations research is indeed a valid one.

Is X possible? Many times the most important contribution mathematicians can make is to demonstrate that an answer is incorrect, a model is unrealistic, an experiment is impossible, or a system is not feasible. Of course, we usually prefer to show that X is indeed possible and how precisely to carry it out.

Finally, can X be optimized? I think we all recognize that many recent contributions of mathematicians have been to optimize, or at least show how to optimize, a solution or system or an experiment or policy.

There are other and more complex questions which come naturally to mind. Typical are questions such as: can X be replaced by Y; or, how sensitive is X to Y; or, how do X and Y trade off? I leave it as a problem for the student to formulate other representative questions! My point is that the kinds of questions or assignments that people in nonacademic mathematics address are rather different from the mathematical questions that academic mathematicians ordinarily entertain. The questions are broader and more complex, less precise, not purely mathematical, and, for this reason, frequently more difficult.

So, what training is required to answer such questions? Obviously, there is no unique and specific pattern of preparation which one could outline. But, let me suggest a few desirable characteristics of the education and training of applied mathematicians for nonacademic pursuits. Consider five which are fairly general and perhaps appropriate for later discussion.

The first I would identify is a background in X. In my experience, the most important single factor for the nonacademic mathematician is a sympathetic interest and a background of understanding in the nonmathematical areas in which he works. This implies a certain attitude on his part which is not usually cultivated in our academic mathematical courses. This attitude, one of unselfish and sympathetic willingness to gain an understanding of what is important in X—and this may not always be what is of greatest mathematical interest in X—needs to be nurtured throughout the education and work experience of applied mathematicians.

The second characteristic is clearly that he should know some mathematics. I say *some* mathematics because I am not one who is persuaded that the mere acquisition of a mass of mathematical facts, theory and abstraction is necessarily paramount. What is most significant is the kind of deep understanding that enables him to know how to use his mathematics, to know when to use it, and, especially, to know when to *avoid* using mathematics. When will simple mathematics do the job, and when is it really necessary or advantageous to use more sophisticated mathematics? He should have some idea of how and where to find the proper tools and methods, how to develop and refine ones that are available, when to look for new methods, and, most important, when to call for help from someone with a completely different background.

One aspect of mathematical training I highly recommend involves well-chosen assignments in mathematical modeling, computation or data analysis and various forms of analytical manipulation. Many times, the kind of problems which people in nonacademic areas must tackle require them to do a considerable amount of analysis, data gathering, statistical analysis and computations, which may not be particularly pleasant but prove most revealing and instructive in arriving at a realistic model or satisfactory answer.

The third aspect is facility in computation—including both numerical analysis and hands-on computing experience. The mathematician needs to develop skill in computing and also skill in avoiding computation. I think these are equally important, and, in fact, I would venture the opinion that more often than not, mathematicians make their most valuable contribution by showing how to *avoid* large scale computations.

The fourth trait is a flair for exposition. Skill in mathematical exposition and also facility for exposition in nonmathematical areas are essential. Many times, the nonacademic mathematician must make clear his results and their implications to a nonmathematical audience (and jury!).

Finally, the nonacademic mathematician should have something that I would characterize as a sense of values or balance or perspective. I mean this in various ways—a sense of values from a personal point of view, so that he is not bothered by the fact that he is not in academia—a sense of perspective from a mathematical point of view, so that he has a realistic appreciation of the worth of the methods and studies which he carries out in comparison to the theoretical papers his academic colleague may well publish—and a balanced view of the relative importance of nonmathematical activities which allows him to appreciate the significance of accomplishments in other fields.

Of course, I do not propose that these traits provide a unique or truly complete characterization for the training of a mathematician, and I expect my colleagues will have much more to say about this. There are other traits to be cultivated—some are intrinsic to the individual. It is good for him to have a certain personal charm. A touch of genius is helpful, also, although too much is not necessarily a good thing. A definite flair for showmanship is undoubtedly most useful for an applied mathematician in nonacademia, and, above all, I have often felt that an independent income is absolutely indispensable!

Edward Hart. Since none of us had any advance way of putting our talks together, there will, of course, be some overlap between what I say and what went before and what will come after. I've chosen to set this up, however, as the answer to a few questions. The questions are finally of this sort. What field of work will a nonacademic mathematician be in? Where will he work? What will he do, more or less specifically? And finally, what should he do about it?

The field of work (to mix a metaphor a bit) is probably some open subset of the whole field of applied mathematics. In the range of industrial applications, I put the General Electric Research and Development Center as somewhere in the middle, not a too restricted place, and yet not like some places where you might find employment as a methematician, such as the National Bureau of Standards. No matter where, your scope will be more limited than in applied mathematics in an academic atmosphere. The range of problems, the time that can be spent on them and the type of approaches that you can apply to them will be more limited. There will be rather considerable overlap between the work of the mathematician and people in other disciplines. With respect to that overlap, I think a distinction should be made right away. I'd like to distinguish between applied mathematicians as such, that is, those who will identify themselves as mathematicians, and those who are in more specialized disciplines that have grown out of mathematics in the last 15 or 20 years. Indeed, part of the problem is due to the fact that the mathematicians, who were brought into industrial and institutional environments in the past twenty years to approach problems for which there were not then established methods of procedure, in a large measure have succeeded, and they have created new disciplines. We call them today computer science, systems engineering, mechanics, statistics, and I'd like to add one more: education, an interesting nonacademic pursuit for a mathematician. I shall concern myself with the mathematician who is not immediately going to identify himself with those fields, and therefore possibly with an academic department that has that name, but instead one who is trying to make it sort of on his own. It is important furthermore to recognize that, being in an industrial and institutional environment, his work will probably be problem or project oriented rather than discipline oriented;

this is one of the things that is connected with the question of attitude that Burt Colvin raised just a while ago.

Now where will he work? Not simply in what company, or where, but in what kind of an environment? In an industrial establishment, he most likely will be attached to an engineering or applications project rather than to some fundamental research unit. That may sound peculiar, but the reason may appear a little later. Parenthetically, let me make a comment that probably has a time limit on it; it has to do with today and perhaps for a few years from now as well. I believe that the prospect for employment—industrial employment—*as a mathematician* is slight. In a more institutional environment—a place like the National Bureau of Standards, or a government or quasi-academic research institute or foundation—a mathematician may find an explicitly mathematical atmosphere in mathematical modeling.

What will he do under such circumstances? While the broad scope of applied mathematics is illustrated by simply looking through several issues of SIAM journals, particularly the applied mathematics journal, the mathematician in industrial or nonacademic applications is going to have more restricted scope than that, and I cannot give any sharp or clear answers. Rather, I shall make a set of comments. First, a negative one: the applied mathematician will be distinguished from the average practitioner in a scientific area by having no experimental orientation or experience. His only advantage over the trained theoretical scientist in that regard will be a superior background in mathematical techniques, if he has them, and if they are needed.

I mentioned before that beyond the specialized areas of mathematics, computer science, etc., the mathematician may find an explicitly mathematical career in modeling. Essential to success in any such direction is a good understanding of the phenomena themselves. That cannot be emphasized too strongly. You cannot, by simply knowing the grammar, write a good novel or do a good study, and to a great extent, mathematics as a discipline is a grammar, it is a way of conceptualizing, but you must know what it is that you are trying to conceptualize. There is still some demand for those skills that we might call "classical analysis". But jobs based on those skills may not be too satisfying in an industrial environment, and their future is somewhat limited; besides, they are to a greater extent being subjected to technological obsolescence by computers. I think it is important that the applied mathematician heading for a nonacademic career identify quite early the specific area in which he intends or expects to work.

Finally, there is no point in attempting to enter a field of work as a professional mathematician unless one is, in fact, a good mathematician. Modern techniques of importance include, at the very least, the unifying approach of modern functional analysis. For many areas of application, you should be well acquainted with methods that employ the techniques developed in differential manifolds and differential forms. Many problems in engineering and scientific application are essentially problems in which one is investigating phenomena that are known in some local region of a space of experience and for which one is trying to determine the behavior outside of that range of experience. Furthermore, numerical methods and theory of approximation are quite important despite the availability of computers. One should not assume, because the orientation in industrial applications will be strongly project or problem oriented, that therefore only the most immediately useable mathematical skills are important. So I emphasize once more that good modern mathematical techniques are of importance for problems today, and they certainly are going to become increasingly important for problems in the future.

In the way of conclusion, then, what should one think of doing? I may not have emphasized strongly enough that in the industrial area the prospects for employment and for activity of the explicitly professional mathematician, who is not associated directly with one of the more specialized fields, are probably slight now. In institutions with a broader base the prospects are better, but—for example—I could probably say that at the General Electric Research and Development Center, no mathematician as such will be hired probably in the next five years, at least. If your area of particular interest and talent is matched by one of the existing organized disciplines that have derived from mathematics, such as mechanics, or systems engineering, then prepare for your career in close association with the appropriate academic department. If you do this in connection with the mathematics department, be sure that you also have close association with the other discipline as well. If you wish to sort of go it on your own, and work in a more general applied mathematics context, then try to associate early with an academic or quasi-academic institution as a prospective employer; in other words, get some idea of where you think you might be going to end up, and then become a good, not too specialized, mathematician.

Alan Hoffman. I disagree in part with what some of my colleagues on this panel have been saying. The point of disagreement concerns the distinctions being made between academic and nonacademic institutions, and academic and nonacademic mathematical research. In both cases, I feel the differences stressed by my colleagues are greater than the differences which exist in practice. Of course, there are some differences in branches of mathematics, and people who work in mechanics know and do things which are technically different from people who work in discrete mathematics; but that difference is independent of the kind of institution where they work. Further, the differences among universities or among laboratories overshadow the differences between the two kinds of institutions. It's true that if you teach in a school you get long summer vacations, and after a while you get tenure and you may feel a little more independent than your colleague in industry. But aside from minor things of that sort, I think the work atmosphere or the kinds of knowledge you need to contribute effectively or to enjoy yourself are very much the same, assuming you are working in the same field.

I think also that we should avoid the successive errors of equating the distinction between academic and nonacademic mathematics with the distinction between pure and applied mathematics, and equating that with the distinction between theory-making and problem solving. My personal experiences, based on 23 years of working in government and industry, make me very skeptical that these equations are valid.

Let me tell you how I view this question of what a mathematician does. In fact, the first time I really had to face this question both of telling myself what a mathematician does and of trying to tell a layman occurred in 1956, when I went to London for the Office of Naval Research. Just before I got off the ship in Southampton, Her Majesty's Immigrations Officer asked me what my occupation was, and I answered, "Mathematician", and he asked me what I did. I said, "Well, I try to analyze situations and try to guess at what is sort of generally true of these situations I'm trying to analyze, and after I make this guess I try to prove that my guess is right." He said, "Are you an accountant?" and I said "No", and then I could see that we were having trouble communicating, as Dean Handelman suggested we might. At that point, I had an inspiration; I said, "You studied geometry at school, didn't you?", and he said "Yes", and I said, "You studied Euclid?", and he said "Yes", and I said, "Well, I'm trying to find out some of the theorems Euclid didn't know." He said, "Oh, that's quite remarkable, we don't have any blokes like that in this country."

The message I'm trying to bring is that "blokes like that" exist both in the academic world and in the industrial world. The zest for mathematical creation or mathematical participation in projects that are both professionally fun, within the discipline, and also fun in the sense of contribution to practical problems in the world, are found in industry as well. I'm not trying to recruit, but I want to convey to you that the atmosphere is not dreary or second-class, it can be perfectly euphoric.

Next, I want to offer two pieces of advice to somebody who is now a student. I hope you won't think it platitudinous: it's based on recollections of what I think I did well and poorly when I was younger. The feeling I have had for some years is this: if you're going to school and you're a student now, don't waste your time—study hard; if you have energy to take another course, do it; if there's a choice between doing less homework and more homework, do the more homework. I can tell you that in these various jobs that I have held at various times, I never found that anything I ever learned as a student was ever lost. Even material from courses in algebraic function theory or in integration in finite terms—which I thought were terribly esoteric—from time to time come up in work. The quality of your work as a scholar, the contribution you make to your organizations and to yourself, and the fun you get out of it, depend partly on your talent and partly on the knowledge you have. The more knowledge you have, the better you are, so don't work as little as you can in order to get by; in fact, work as hard as you can without driving yourself batty. That's one piece of advice.

The second advice is to remind you that skill in teaching is really necessary in industrial jobs. As you probably also know, a fair share of the work of a scholar in industry is in the role of teacher. He lectures, he makes presentations, he works with colleagues; his role as teacher in many respects is just as direct as that of the professor at school, and is just as challenging. If any of you are in a position where you are asked if you would like to teach a course, and you have a chance to say yes or no, by all means say yes. Quite apart from whether or not you get paid, it is just a very useful thing to have done. It improves you in every possible way. Teach at every opportunity, provided you have the time.

I turn to the question of whether you are better off learning this or that. Something that is surprising and gratifying is how easily people who have skill and are well trained in mathematics can adapt. For example, one of Burt Colvin's colleagues is a man who as a graduate student was totally abstract; none of his research or course interests as a graduate student had any contact with reality at all. His present work is very practical and very good. Professor Seidel and I have a friend who is a very distinguished academic combinatorial mathematician. He has also done very successful practical work on certain combinatorial problems of importance in the design of communication networks. A lot of this adaptability depends on whether you're capable and interested and enthusiastic, and on how much you know in general. These are things that count more than any particular kind of course training. Of course, in this respect I address myself to people who intend to make careers in research mathematics. To those who don't have such intentions, I think one could make more formal statements about the kind of training that you're more likely to find useful industrially, as well as academically.

The last remark I want to make is this: right now there are very few industrial jobs; for that matter, there are very few academic jobs, but there are fortunately many more than industrial jobs. I can't see why anyone now a student should say to himself, "I'm more interested in an industrial than an academic job, and therefore I'm going to study this, this and this because that prepares me more for a research industrial career than if I were to study this, this and this, which would prepare me for an academic career." I think that teaching is a noble profession and it's perfectly sensible to prepare yourself for that. You may not get a job at a university, you may go to work industrially, but the best thing to do is to study as proficiently and enthusiastically as you can in the things that interest you, and find later where the employment opportunities are.

It's so hard to give advice at large rather than to a particular person, but in general this is my current feeling about the kind of counsel that might be offered to students generally. Thank you very much for listening.

Murray Klamkin. First, it should be noted that what is being voiced here has been voiced repeatedly over the last few generations and, unfortunately, without too much effect. Perhaps, due to the present economic climate, this meeting and others like it may begin to make a real mark on the educational system. In particular, I refer to two previous SIAM conferences. First, "Applied Mathematics; What is Needed in Research and Education", which was published in *SIAM Review*[3] in October 1962. Second, "Education in Applied Mathematics", published in *SIAM Review*[4] in April 1967. Incidentally, Professor Carrier, who was to be our moderator, and also Burt Colvin were participants at these particular conferences. Another important one was held at Brown recently on "The Future of Applied Mathematics", and this was published in *Quarterly of Applied Mathematics*[5] in April 1972. If you haven't already done so, I suggest very strongly that you read these with respect to the topic being discussed here.

As an indication of some of the criticisms of mathematics and its teaching, I'll start out with a story told at the first SIAM conference by C. N. Yang, who relates the feeling a physicist has when he consults a mathematician. There was a man who had a big pile of dirty laundry, and he finally was relieved when he found

[3] This Review, 4 (1962), pp. 297–320.
[4] This Review, 9 (1967), pp. 289–415.
[5] Quart. Appl. Math., 30 (1972), pp. 1–125.

a store with a sign saying, "Laundry Done Here". He goes inside, dumps his laundry on the counter. The man looks up from the counter and says, "What's all this?" The man says, "I want my laundry done." "Oh, we don't do laundry here." "What do you mean you don't do laundry here? You've got that sign in the window." "Oh, that—we just make signs."

At that same conference, Professor Carrier referred to a definition of Joe Keller on what applied mathematics is. Joe Keller's definition was: applied mathematics is that science of which pure mathematics is a branch. Although this is a sort of tongue-in-cheek definition, it does make the point that to do applied mathematics you have to know things other than mathematics. Related to this, Sydney Goldstein once said that to be a good applied mathematician you have to be one-third physicist, one-third engineer and one-third mathematician.

A much better description of applied mathematics has been given by Professor Synge, who's presently senior professor at the Dublin Institute of Advanced Study. He notes that applying mathematics to a real problem involves three stages. The first stage is a dive from the world of reality into the world of mathematics. The second stage is a swim in the world of mathematics. The third stage is a climb back from the world of mathematics into the world of reality, and, most importantly, with a prediction in your teeth. With respect to these three stages, one of the criticicisms of mathematics teaching is that too much time is paid exclusively to the middle stage, the swim in the ocean of mathematics. This is certainly important, but so far as applications of mathematics in the real world are concerned, the first and third stages are equally important. Unless we stress these as much as the second stage, we're going to have difficulties.

Other mathematicians such as Sutton, Krank and Pollak have amplified these three stages into five or more stages. In terms of the one-word descriptions of Henry Pollak, we first have *recognition*. First of all you have to recognize that there is a problem, and what the problem is. Two: once the problem has been recognized, there must be a mathematical *formulation* of the problem. After the formulation stage is done, you have to get a mathematical *solution*. Usually the mathematical solution involves a certain amount of computation. That's the fourth stage: *computation*. Coupled with the last two stages, we have to have feedback to the original first stage of recognition to make sure that the problem we have solved is indeed the problem that we really wanted to solve and not some other one that is related to it. And five: we have the all-important last stage, which has been voiced repeatedly: *communication*.

In regard to the first two stages of recognition and formulation, Synge back in 1944 in *American Mathematical Monthly*[6] had this to say about the purist mathematicians: "Nature will throw out mighty problems but they will never reach the mathematician. He will sit in his ivory tower waiting for the enemy with an arsenal of guns, but the enemy will never come to him. Nature does not offer her problems ready formulated. They must be dug up by pick and shovel, and he who will not soil his hands will never see them."

There's a big difference in attitude between an applied and a pure mathematician. For instance, a pure mathematician can work on a given problem for

[6] J. L. Synge, *Focal properties of optical and electromagnetic systems*, Amer. Math. Monthly, 51, (1944), pp. 185–200.

years, and if it turns out to be intractable, he can alter the problem. To paraphrase a line from *Finian's Rainbow*, if a mathematician cannot handle the problem on hand, he'll handle the problem out of hand. The applied mathematician when he is given a problem must handle the particular problem on hand; his job may depend on it. First of all, he'll have to check to see if the problem that's given to him is really the problem that they're trying to solve, because many times the problem they give to you is not really the problem they want. He must come up with an answer, usually an approximation, that is optimal, in a sense, with regard to cost and time. You can't work on the problem for five years unless it's a very important problem to the company, and usually they're not that important. Finally, if he's successful he must communicate his results in such a way that they're understandable to his superiors. In this regard, I'd like just to mention something about mathematical papers that are published. Mark Kac once estimated that the expected number of people to read any mathematical paper is three: the author, the referee and the reviewer. Some other editors have facetiously estimated the number to be two: the reviewer and the referee.

I will now devote the rest of my time to some problems. If I had more time, I could go into these problems more deeply with regard to some of my previous remarks. Consider the problem—which is not a new one—of finding the best location of a common school to serve three small neighboring towns that can't afford to build three schools. The problem here is not yet a mathematical one. The first question to resolve is what do you mean by best? In other words, what's the norm in this particular problem? This is still not a mathematical problem; this is a combination of a political, sociological and economic problem. Unless you can first get a grip on this problem which is outside of mathematics, there won't be any mathematical problem to consider. Of course, one particular norm is that you can say that you don't want any child to travel any further than any other child; assuming that the towns can be viewed as points, then the solution would be the center of the circumscribed circle about the three towns. If you want to minimize the cost of the road, then you have the Steiner problem; if you want to minimize the number of children-miles traveled today, then you have the weighted Steiner problem. Thus, the first big problem here is to determine the appropriate norm.

Another problem: consider airline companies. You want to fly from, say, Albany to California. Big problem: you want to minimize cost. Usually the airplanes cruise at a fixed speed, so the problem would be how should you fly to get there in the least time. We're assuming there are wind currents. If there weren't any wind currents, you would fly an essentially straight path. Now the straight path minimizes the distance with respect to ground but it doesn't minimize the time. Many years ago, in the 1920's there was a paper by McShane who considered the problem due to Zermelo, called the Zermelo navigation problem. He was given a wind field which varies both with time and space and an airplane flying at a constant speed with respect to the ground. The problem was to find the minimal time path. McShane proved, under sufficient smoothness conditions of the wind field, that a solution exists. So far as the airlines are concerned, they want to know how should one fly the airplane, and that is an entirely different problem and a much more difficult one. At present, the airlines do have an algorithm for the flight path which takes into consideration, as well, more realistic and complicated things. For example, it takes into account the variations in barometric pressure, gravity and a constraint of avoiding a too bumpy ride due to turbulence.

Another problem, that I call the Fresnel lens problem, I solved for my company about a year and a half ago. If you look at the brake lights on your car you will see red lenses behind them. The government decided that whatever the light intensity was at various points, it had to be considerably increased; they figured it would be safer. Now one simple solution to this problem is just to increase the candlepower output of the bulb, but that's not an efficient solution because the battery is already loaded with lots of peripheral equipment, and they wanted to do a better job. The optical engineers redesigned the Fresnel lens to do a better job of concentrating the light, and then the problem was to make a die to make this particular lens. Now Ford doesn't make everything. They gave the design of this lens to an outside vendor to make the die from which you then make plastic lenses. The outside vendor came back and said, "Will you relax the tolerances for the inner groove because we cannot make it?" The optical people weren't sure why they couldn't make it; one had heard me give a lecture in solid geometry and figured this was a geometric problem, so he finally came to me and said, "Why can't these people turn this groove with our tolerance?" And I must say I didn't know. In fact, I had to dirty my hands for a couple of days because it's an interesting problem in three-dimensional geometry, essentially only analytic geometry, but not of the normal sort. One of my criticisms here is that although geometry crops up frequently in applications, not much appropriate geometry is being given in our schools. The late Professor Steenrod had taught elementary calculus courses to see what they involved. He came to the conclusion that in calculus there's lots of geometry, but unfortunately the students get little of it because due to limitations of time and everything else, the instructor essentially does all the geometry and leaves just the strict analysis for the student. Then when they get the answers, the reinterpretation back into geometric form is again done by the instructor. When I worked for AVCO, I found the biggest failing in the engineers and scientists setting up problems was their rudimentary knowledge of geometry.[7]

The next problem will connect with a remark made by Burt Colvin. It is the problem of a brake cable. Apparently on some model cars, there was trouble with the brake cable being hit by the tire when you make a sharp turn, and eventually this could wear the brake cable and you would lose your brakes. The engineers wanted to have a simple way of knowing what the shape of the brake cable would be for a given length so they could adequately design for it. This is the problem of a three-dimensional tortuous elastica. You are given a flexible cable that is attached to two fixed points at fixed slopes, and you wish to find what the shape is knowing the length of the cable and these boundary conditions. Unfortunately, this involves three simultaneous second order nonlinear differential equations with four unknown constants and six boundary conditions. Now, even today, nobody has come up with a good mathematical scheme to do this problem

[7] M. S. Klamkin, *On the ideal role of an industrial mathematician and its educational implications*, Amer. Math. Monthly, 78, (1971), pp. 53–76.

even with a computer. As far as I was concerned, I said, "Look, just change the length and see what happens; when it clears, that's it." So you have to have a sense of proportion as was mentioned previously. There are certain times when the mathematics is too complicated, and besides you always have to go back and check to see if your model is realistic.

Since my time is getting short, let me finish up with just one more problem. This is sort of a purist problem, but it does indicate again a difference in attitude between a pure mathematician and an applied mathematician. This is an old problem. Let's say we have a rectangular table and we wish to play a gambling game. We place down dimes alternately—me, somebody else—the coins cannot overlap the table or overlap another coin. The person who puts down the last coin collects them all. The problem is: how should you play? Since I give you the problem, I should at least give you a choice. You can go first or second; which would you prefer to do? One problem solving strategy, which is not apparently widely used, is the one of continuity. It you have a problem that you're in doubt about, change the data and see what happens. If necessary, change the data strongly. If we were to play this game on a big table it would take a long time. Let's say we have dimes that are three-quarters of an inch in diameter, and let's play on a table that is three-quarters of an inch by three quarters of an inch. Only the first person can move; therefore it's a tie and nobody is going to lose. Now we'll play with a table two units by one unit. Again this game will be a tie because the first person can move in the middle and then the second person can't move. Otherwise, if the first person moved to the edge, he would lose. If you now play on a table three by one, it pays to go first. You put your first move in the center. Regardless of where the other man moves, you still can move. Consequently, the solution of this problem is: given any centro-symmetric table, the first move is right in the middle of the table and then successive moves are centro-symmetric to the opponent. This is the classical solution. However, you should always reconsider your solution by asking, "Is this a well-posed or ill-posed problem?" In applied mathematics and engineering, we say a well-posed problem is one whose solution depends continuously on the data. But I submit that this is an ill-posed problem. Can you align the coins exactly centro-symmetrically? It may be that your perfect strategy may be exactly the wrong strategy in practice. So your apparent winning solution can be a losing solution. I repeat, you should always go back and see whether your problem is well-posed. You may have a beautiful mathematical solution, but due to actual conditions, there may be something which makes the problem ill-posed.

SUPPLEMENTARY REFERENCES

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[3] R. E. Gaskell and M. S. Klamkin, *The industrial mathematician views his profession*, Amer. Math. Monthly, 81 (1974), pp. 699–716.

[4] R. R. McLone, *The Training of Mathematicians, A Research Report*, Social Science Research Council, London, 1973.

[5] C. F. A. BEAUMONT, *The Study of Mathematical Sciences in Canada*, University of Waterloo, Ontario, Canada, 1974.
[6] T. SEIDMAN, *A proposal for a professional program in mathematics*, Amer. Math. Monthly, to appear.

J. J. Seidel. I would like to add a few remarks in describing a curriculum that we have in our technological university in Eindhoven in the Netherlands that we call a curriculum in mathematical engineering. I didn't hear this word here; I just heard applied mathematics. I don't like that term so very much for several reasons. One reason is that almost always people think of applied analysis, whereas in these days I think algebra and set theory are also fields that may be useful. On the other hand, I also don't like a controversy between pure and applied mathematics, and really I don't understand very well what that means. I shall tell you what we use as a definition for mathematical engineering. The domain of the mathematical engineer is the approach with mathematical tools to problems of technology, management, physical and information sciences. The word engineer indicates the type of problems involved, and the word mathematical refers to the methods of investigation. The combination of these words reflects the union of method and problem which is essential for the philosophy behind this program. The program covers five years, and at the end, the student is called a mathematical engineer and is recognized in our country as such.

I shall not go into detail in describing to you this program. In the first part, apart from the ordinary mathematics for the engineer, there is a fundamental course, a course in fundamental mathematics, and introductions to statistics, mechanics, the art of programming, numerical analysis and discrete mathematics. Of course there is also some amount of physics and of electronics, but there is no course, for instance, in topology as such. We do teach topology, but it is introduced whenever it is needed as a tool, and not as a special subject. Analysis is oriented toward hard analysis, toward problem solving. We are, finally, allergic to general abstract nonsense. One problem we have in this beginning stage is how to reflect at an early stage of the study the mathematical models that will become important later. That really is our problem.

I'll describe to you also the studies at the end of the five-year program. At the end, apart from more specialized courses of lectures and seminars, the main task of the student is to work for about nine months on a real problem on which he must report in a kind of master's thesis. We now have some seventy-five such theses available. The thesis may concern the solution of physical or technical problems with analytic methods, the evaluation of data from mathematical models of a stochastic nature, mathematical and numerical aspects of problems for decision and control, automatic manipulation of numerical and nonnumerical data, the theory of programming, the construction of software, problems of graph theory and error-correcting codes, and also problems of a more abstract nature. Sometimes the problems come from other engineers, from other engineering departments or from industry. The use of computers is almost always an essential tool.

I would like to point out that this program is a rather young program, although it is taught now at all three technological universities in our country. It is a bit too early really to give reliable statistics about what happens to these people, but let me end by just saying that, contrary to the other speakers, my job is at the university, so I have no need to express any pessimism in the future of the mathematical engineer.

3. Summary of discussion period (by William E. Boyce). During the discussion period, questions and comments centered about three principal concerns:

(a) Employment of mathematicians in industry and government;
(b) Aspects of the educational process, such as courses and texts;
(c) Interaction between the academic world and government or industry.

Employment of mathematicians in industry and government. Colvin stated that there is still a relatively good opportunity to earn a living as a mathematician if one's mathematical training is combined with a good foundation in another field. Hart emphasized the importance of being strongly identified with the field of application. He also reiterated that if one wishes to make a contribution as a mathematician, it is essential to be a *good* mathematician. People with mathematical training will be hired, but probably not as mathematicians. Once hired, people will be labeled not so much according to discipline as according to job function.

Hoffman indicated that IBM would continue to hire a few mathematicians, persons with a background and degrees in mathematics. He said again that he saw no sharp distinction between pure and applied mathematics, or between academic and industrial mathematics, and urged that one should study intensively whatever one feels a calling for, letting the question of employment come later. A member of the audience said that five graduating mathematics majors (out of a class of 15) from his institution had recently been hired by IBM, but as systems engineers.

Klamkin expressed pessimism because the increasing mathematical competence of many scientists and engineers and the rapid improvement of computing facilities tend to reduce the need for mathematical consultants in industry. Colvin agreed that many tasks necessarily carried out by mathematicians in the past are now done by people with other backgrounds. He suggested that young applied mathematicians would do well to avoid fields, such as physics and engineering, that are highly developed, and instead should look at areas such as economics, banking, biology, medicine, psychology and sociology. As an example, he mentioned a staff member at NBS who is currently working for the state of Washington on the development of a mathematical model of salmon hatcheries.

Colvin further noted a development that many people thought would take place in the period from 1950 to 1970, but which has not occurred. It was widely expected that by this time there would be many mathematicians in industry or government who would be in a position to influence hiring policies. In fact, this is not the case. As a result, job descriptions are usually framed by persons with other backgrounds, who tend to describe the jobs in nonmathematical terms. Thus, even if the nature of the work is largely mathematical, this fact may be obscured.

A question was raised about whether it is fair to try to attract women to applied mathematics, in view of the present employment situation. Responses were inconclusive. Colvin noted that women make up about one-quarter of the professional staff in his division at NBS, including one supervisor.

Aspects of the educational process. In response to a question about text materials at an elementary level, Klamkin indicated that the book[8] by Ben Noble is a good first approximation, but that an instructor should expect to supplement it. Klamkin also mentioned the existence of a new section in *SIAM Review*, called "Classroom Notes in Applied Mathematics", which will contain short and essentially self-contained notes on applications that may be useful to instructors.

A question was raised as to what elementary course in applied mathematics should be offered, if only one such course can be given. The responses were operations research and linear programming (Hoffman) and classical mechanics (Handelman). In a somewhat broader vein, Colvin proposed a seminar possibly involving several instructors and ranging over a variety of topics; the level of the seminar could be adjusted to the needs and backgrounds of the participants.

Following a question about possible changes in present-day practices in mathematics education that would enhance a graduate's qualifications as viewed from industry, Seidel indicated that perhaps a greater emphasis on problem solving would be useful. It was generally agreed that experience in model building should begin early, preferably at the level of secondary or even elementary school. Hoffman observed that short of the doctoral level, by far the largest markets for mathematics students are as teachers or programmers. He pointed out that it was absurd for an undergraduate student not to learn programming. At the M.S. level, the student's qualifications can be improved by a knowledge of statistics, operations research and numerical analysis.

Interaction between the academic world and government or industry. A question was raised as to how to promote such interaction, especially with reference to (a) sabbatical leaves for faculty, (b) summer jobs for students.

The members of the panel generally agreed that such programs were desirable, and that each case would be handled individually. Klamkin pointed out that at Oxford there are periodic seminars in which industrial mathematicians are brought to the campus for discussion of current problems. This sometimes leads to consulting opportunities for the faculty or jobs for students. He stated that such seminars could well be held at many campuses in this country.

Colvin stated that many government installations and industrial laboratories have programs of summer employment for faculty members and students. He cautioned that the selection of personnel and their assignments would be guided primarily by the need to accomplish certain tasks, and that summer employment should not be viewed as a program to train individuals.

Hoffman mentioned that it would be desirable for industrial internships to take place at times other than the summer, or for longer periods of time, if maximum benefits are to be obtained.

[8] Ben Noble, *Applications of Undergraduate Mathematics in Engineering*, MacMillan, New York, 1967.